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Revisiting Quaternion Dual Electrodynamics

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Abstract Dual electrodynamics and corresponding Maxwell's equations (in the presence of monopole only) are revisited from dual symmetry and accordingly the quaternionic reformulation of field equations and equation of motion is developed in simple, compact and consistent manner.

Keywords Quaternion · Monopole · Dual · Electrodynamics

1 Introduction

Quaternions were invented by Hamilton [1, 2] in 1843 and Tait [3, 4] promoted them in order to solve the problems of mathematics and physics. Quaternions represent the natural extension of complex numbers and form an algebra under addition and multiplication. The reasons why Hamilton's quaternions have never become a prevalent formalism in physics, while Hamilton's formulation of complex numbers has been universally adopted, is an open question. The fact is that 'quaternion structures' are very frequent in numerous areas of physics, the most prominent examples being special relativity (i.e., Lorentz transformations), electrodynamics, and spin. For this reason quaternions and their generalizations keep reappearing in a number of forms, which are as numerous as diverse: spinors, Einstein's semi vectors, Pauli matrices, Stokes parameters, Eddington numbers, Clifford numbers, qubits, etc. On the other hand, because quaternion algebra yields more efficient algorithms than matrix algebra for three and four dimensional applications, their use in computer simulations and

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graphics, numerical and symbolic calculations, robotics, navigation, etc., has become more and more frequent in the past few decades. A comprehensive list of references has already been published in quaternion bibliographies [5] on the practical applications of quaternion analysis on various applied topics in theoretical and mathematical physics. The quaternionic formulation of electrodynamics has a long history [6–16], stretching back to Maxwell himself [6–8] who used real (Hamilton) quaternion in his original manuscript 'on the application of quaternion to electromagnetism' and in his celebrated book "Treatise on Electricity and Magnetism". However, Maxwell [6–8] used quaternion as the substitute of common vector calculus which made his field equations difficult for his contemporaries [17, 18] because the quaternionic formulation in three-space brings several complications as the field of applicability of real quaternions is Euclidean four-space [19–23]. Therefore, the turning point, in using quaternions in theoretical physics, was the creation of special relativity which unites space and time forming a four-dimensional space-time. The formulation of physical laws using real quaternions has then be replaced by complex ones, and it has been recognized that complex quaternions represent a powerful instrument in formulating classical physical laws. Complex (bi-) quaternions form a division ring having a number of desirable properties that allow the powerful theorems of modern algebra to be applied. Quaternion analysis has since been rediscovered at regular intervals and accordingly the Maxwell's Equations of electromagnetism were rewritten as one quaternion equations [24–39]. We have also [40–46] studied the quaternionic formulation for generalized electromagnetic fields of dyons (particles carrying simultaneous existence of electric and magnetic charges) in unique, simpler and compact notations.

Applying the electromagnetic duality in Maxwell's equation, in this paper, we have discussed the dual electro dynamics, dual Maxwell's equation, and equation of motion for dual electric charge (i.e. for magnetic monopole). We have also reformulated the dynamical equations of dual electrodynamics in terms of quaternion variables and accordingly developed the dual quaternion electrodynamics in simple, compact and consistent manner. It has been concluded that quaternionic formulation, where the field equations reduce to a single quaternion equation, has closed analogy between the sub and superluminal objects as the norm of a quaternion four-vector behaves in the same manner as it does under the influence of superluminal Lorentz transformations.

2 Dual Electrodynamics

The concept of electromagnetic (EM) duality has been receiving much attention [47–49] in gauge theories, field theories, supersymmetry and super strings. Duality invariance is an old idea introduced a century ago in classical electromagnetism for the following Maxwell's equations in vacuum (using natural units $c = \hbar = 1$, space-time four-vector $\{x^\mu\} = (t, x, y, z)$, $\{x_\mu\} = \eta_{\mu\nu} x^\mu$ and $\{\eta_{\mu\nu} = +1, -1, -1, -1 = \eta^{\mu\nu}\}$ through out the text),

$$\begin{aligned} &\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0, \qquad \overrightarrow{\nabla} \cdot \overrightarrow{H} = 0, \\ &\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{H}}{\partial t}, \qquad \overrightarrow{\nabla} \times \overrightarrow{H} = \frac{\partial \overrightarrow{E}}{\partial t}, \end{aligned} \tag{1}$$

as these were invariant not only under Lorentz and conformal transformations but also invariant under the following duality transformations,

$$\begin{aligned} \overrightarrow{E} &\Rightarrow \overrightarrow{E} \cos\vartheta + \overrightarrow{H} \sin\vartheta, \\ \overrightarrow{H} &\Rightarrow -\overrightarrow{E} \sin\vartheta + \overrightarrow{H} \cos\vartheta, \end{aligned} \tag{2}$$

where $\overrightarrow{E}$ and $\overrightarrow{H}$ are respectively the electric and magnetic fields. For a particular value of $\vartheta = \frac{\pi}{2}$, (2) reduces to

$$\overrightarrow{E} \to \overrightarrow{H}, \qquad \overrightarrow{H} \to -\overrightarrow{E}, \tag{3}$$

which can be written as

$$\begin{pmatrix} \overrightarrow{E} \\ \overrightarrow{H} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{E} \\ \overrightarrow{H} \end{pmatrix}. \tag{4}$$

Let us introduce a complex vector $\overrightarrow{\Psi} = \overrightarrow{E} + i\overrightarrow{H}$ ($i = \sqrt{-1}$) so that the Maxwell's equations (1) be written as

$$\overrightarrow{\nabla} \cdot \overrightarrow{\Psi} = 0, \qquad \overrightarrow{\nabla} \times \overrightarrow{\Psi} = i \frac{\partial \overrightarrow{\Psi}}{\partial t}, \tag{5}$$

which is also invariant under following duality transformations

$$\overrightarrow{\Psi} \to \exp(i\vartheta)\overrightarrow{\Psi}. \tag{6}$$

The duality symmetry is lost if electric charge and current source densities enter to the conventional Maxwell's equations given by

$$\begin{aligned} &\overrightarrow{\nabla} \cdot \overrightarrow{E} = \rho, \qquad \overrightarrow{\nabla} \times \overrightarrow{H} = \overrightarrow{j} + \frac{\partial \overrightarrow{E}}{\partial t} \quad \Longrightarrow \quad \partial_\nu F^{\mu\nu} = j^\mu, \\ &\overrightarrow{\nabla} \cdot \overrightarrow{H} = 0, \qquad \nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{H}}{\partial t} \quad \Longrightarrow \quad \partial_\nu \widetilde{F^{\mu\nu}} = 0, \end{aligned} \tag{7}$$

where $\{j^\mu\} = (\rho, \overrightarrow{j})$ is described as four-current source density. Consequently, Maxwell's equations may be solved by introducing the concept of vector potential in either two ways [50–54]. The conventional choice has been used as

$$\overrightarrow{E} = -\frac{\partial \overrightarrow{A}}{\partial t} - \operatorname{grad}\phi, \qquad \overrightarrow{H} = \overrightarrow{\nabla} \times \overrightarrow{A}, \tag{8}$$

where $\{A^\mu\} = (\phi, \overrightarrow{A})$ is called the four potential and accordingly the second pair (i.e., $\overrightarrow{\nabla} \cdot \overrightarrow{H} = 0$; $\nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{H}}{\partial t} \Longrightarrow \partial_\nu \widetilde{F^{\mu\nu}} = 0$) of the Maxwell's equations (7) become kinematical identities and the dynamics is contained in the first pair (i.e., $\overrightarrow{\nabla} \cdot \overrightarrow{E} = \rho$; $\nabla \times \overrightarrow{H} = \overrightarrow{j} + \frac{\partial \overrightarrow{E}}{\partial t} \Longrightarrow \partial_\nu F^{\mu\nu} = j^\mu$) and $\Box A^\mu = j^\mu$ with the D'Alembertian $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$. Here, we have used the definition $F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$ for anti-symmetric electromagnetic field tensor whose components are $F^{0j} = E_j$; $F^{jk} = \varepsilon^{jkl} H_l$ ($\forall j, k, l = 1, 2, 3$), $\varepsilon^{jkl} = +1$ for cyclic, $\varepsilon^{jkl} = -1$ for anti-cyclic permutations, $\varepsilon^{jkl} = 0$ for repeated indices; $\widetilde{F^{\mu\nu}} = \frac{1}{2}\epsilon^{\mu\nu\lambda\omega} F_{\lambda\omega}$ ($\forall \mu, \nu, \eta, \lambda = 0, 1, 2, 3$); being the dual electromagnetic field tensor with $\widetilde{F^{0j}} = H_j$ and $\widetilde{F^{jk}} = \varepsilon^{jkl} E_l$; $\epsilon^{\mu\nu\lambda\omega}$ as the four dimensional generalization of ε^{jkl}. Equation (5) is now modified as

$$\overrightarrow{\nabla} \cdot \overrightarrow{\Psi} = \rho, \qquad \overrightarrow{\nabla} \times \overrightarrow{\Psi} = i \frac{\partial \overrightarrow{\Psi}}{\partial t} + i\overrightarrow{j}, \tag{9}$$

which is no more invariant under duality transformations (6). Here if we consider the another alternative way to write

$$\overrightarrow{H} = -\frac{\partial \overrightarrow{B}}{\partial t} - \operatorname{grad}\varphi, \qquad \overrightarrow{E} = -\overrightarrow{\nabla} \times \overrightarrow{B}, \tag{10}$$

by introducing another potential $\{B^{\mu}\} = (\varphi, \overrightarrow{B})$, we see that source free Maxwell's equations (1) retain their forms but Maxwell's equations (7) reduce to

$$\begin{aligned} \overrightarrow{\nabla} \cdot \overrightarrow{E} &= 0, \qquad \overrightarrow{\nabla} \times \overrightarrow{H} = \frac{\partial \overrightarrow{E}}{\partial t}, \\ \overrightarrow{\nabla} \cdot \overrightarrow{H} &= \sigma, \qquad \nabla \times \overrightarrow{E} = -\overrightarrow{\kappa} - \frac{\partial \overrightarrow{H}}{\partial t}, \end{aligned} \tag{11}$$

where the first pair ($\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0$; $\overrightarrow{\nabla} \times \overrightarrow{H} = \frac{\partial \overrightarrow{E}}{\partial t}$) becomes kinematical while the dynamics is contained in the second pair ($\overrightarrow{\nabla} \cdot \overrightarrow{H} = \sigma$; $\nabla \times \overrightarrow{E} = -\overrightarrow{\kappa} - \frac{\partial \overrightarrow{H}}{\partial t}$). Equation (11) may then be written in following covariant forms

$$\begin{aligned} \partial_{\nu} F^{\mu\nu} &= 0 \quad \text{or} \quad F_{\mu\nu,\nu} = 0, \\ \partial_{\nu} \widetilde{F^{\mu\nu}} &= k^{\mu} \quad \text{or} \quad \widetilde{F_{\mu\nu,\nu}} = k_{\mu}, \end{aligned} \tag{12}$$

where $\widetilde{F^{\mu\nu}} = \partial^{\nu} B^{\mu} - \partial^{\mu} B^{\nu}$; $\widetilde{\widetilde{F^{\mu\nu}}} = F^{\mu\nu}$; $\{k^{\mu}\} = (\sigma, \overrightarrow{\kappa})$ and $\{k_{\mu}\} = (\sigma, -\overrightarrow{\kappa})$. Equation (11) may also be obtained if we apply the transformations (3) and (4) along with the following duality transformations for potential and current i.e.

$$\begin{aligned} A^{\mu} \to B^{\mu}, \quad B^{\mu} \to -A^{\mu} &\Longleftrightarrow \begin{pmatrix} A^{\mu} \\ B^{\mu} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A^{\mu} \\ B^{\mu} \end{pmatrix}, \\ j^{\mu} \to k^{\mu}, \quad k^{\mu} \to -j^{\mu} &\Longleftrightarrow \begin{pmatrix} j^{\mu} \\ k^{\mu} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} j^{\mu} \\ k^{\mu} \end{pmatrix}. \end{aligned} \tag{13}$$

So, we may identify the potential $\{B^{\mu}\} = (\varphi, \overrightarrow{B})$ as the dual potential and the current $\{k^{\mu}\} = (\sigma, \overrightarrow{\kappa})$ as the dual current. Correspondingly the differential equations (11) are identified as the dual Maxwell's equations and we may accordingly develop the dual electrodynamics. Introducing the electromagnetic duality, in the Maxwell's equations, we may establish the connection between electric and magnetic charge (monopole) [55–60], with the fact that an electric charge interacts with its electric field as the dual charge (magnetic monopole) interacts with magnetic field, as,

$$e \to g, \quad g \to -e \Longleftrightarrow \begin{pmatrix} e \\ g \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix}, \tag{14}$$

where g is described as the dual electric charge (charge of magnetic monopole). Hence we may recall the dual electrodynamics as the dynamics of pure magnetic monopole and the corresponding physical variables associated there are described as the dynamical quantities of magnetic monopole. As such, we write the duality transformation for $F^{\mu\nu}$ and $\widetilde{F^{\mu\nu}}$ as

$$F^{\mu\nu} \to \widetilde{F^{\mu\nu}}, \quad \widetilde{F^{\mu\nu}} \to -F^{\mu\nu} \Longleftrightarrow \begin{pmatrix} F^{\mu\nu} \\ \widetilde{F^{\mu\nu}} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} F^{\mu\nu} \\ \widetilde{F^{\mu\nu}} \end{pmatrix}. \tag{15}$$

So, we rewrite the dual Maxwell's equations (11) as the field equations for magnetic monopole (or in the absence of electric charge) on replacing σ by magnetic charge density ρ_m and $\overrightarrow{\kappa}$ by the magnetic current density $\overrightarrow{j_m}$ as

$$\begin{aligned} \overrightarrow{\nabla} \cdot \overrightarrow{E} &= 0, \\ \overrightarrow{\nabla} \cdot \overrightarrow{H} &= \rho_m, \\ \overrightarrow{\nabla} \times \overrightarrow{E} &= -\frac{\partial \overrightarrow{H}}{\partial t} - \overrightarrow{j_m}, \\ \overrightarrow{\nabla} \times \overrightarrow{H} &= \frac{\partial \overrightarrow{E}}{\partial t}. \end{aligned} \tag{16}$$

The complex vector field $\overrightarrow{\Psi} = \overrightarrow{E} + i\overrightarrow{H}$ $(i = \sqrt{-1})$ is now replaced by $\overrightarrow{\psi}$ as the consequence of duality (3) and (4) i.e.

$$\overrightarrow{\psi} = \overrightarrow{H} - i\overrightarrow{E}. \tag{17}$$

Substituting the $\overrightarrow{E}$ and $\overrightarrow{H}$ from (10) on replacing φ by ϕ_m we may establish the following relation between the electromagnetic vector $\overrightarrow{\psi}$ and the components of the magnetic four-potential as,

$$\overrightarrow{\psi} = -\overrightarrow{\nabla}\phi_m - \frac{\partial \overrightarrow{B}}{\partial t} + i\overrightarrow{\nabla} \times \overrightarrow{B}. \tag{18}$$

Hence we may write the Maxwell's equations (16) for monopole in terms of complex vector field $\overrightarrow{\psi}$ as

$$\begin{aligned} \overrightarrow{\nabla} \cdot \overrightarrow{\psi} &= \rho_m, \\ \overrightarrow{\nabla} \times \overrightarrow{\psi} &= i\overrightarrow{j_m} + i\frac{\partial \overrightarrow{\psi}}{\partial t}. \end{aligned} \tag{19}$$

Accordingly let us replace dual $\widetilde{F^{\mu\nu}}$ by the field tensor $\mathcal{F}^{\mu\nu}$ for magnetic monopole as,

$$\mathcal{F}_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu \quad (\mu, \nu = 1, 2, 3), \tag{20}$$

which reproduces the following definition of magneto-electric fields of monopole as

$$\begin{aligned} \mathcal{F}_{0i} &= H^i, \\ \mathcal{F}_{ij} &= -\varepsilon_{ijk}E^k. \end{aligned} \tag{21}$$

Hence the covariant form of Maxwell's equations (12) for magnetic monopole may now be written as

$$\begin{aligned} \mathcal{F}_{\mu\nu,\nu} &= k_\mu, \\ \widetilde{\mathcal{F}_{\mu\nu,\nu}} &= 0, \end{aligned} \tag{22}$$

where $\{k_\mu\} = \{\rho_m, -\overrightarrow{j_m}\}$ is the four-current density of the magnetic charge. Now using (10) and (16), we get

$$\begin{aligned} \square\phi_m &= \rho_m, \\ \square\overrightarrow{B} &= \overrightarrow{j_m}, \end{aligned} \tag{23}$$

where we have imposed the following Lorentz gauge condition

$$\overrightarrow{\nabla}\cdot\overrightarrow{B} + \frac{\partial\phi_m}{\partial t} = 0. \tag{24}$$

Equation (23) can also be generalized in the following covariant form, which may directly be obtained from (22), as

$$\square B_\mu = k_\mu. \tag{25}$$

Consequently, the divergence of the third equation (16) (i.e. div of $[\overrightarrow{\nabla}\times\overrightarrow{E} = -\frac{\partial\overrightarrow{H}}{\partial t} - \overrightarrow{j_m}]$) leads to the following continuity equation of magnetic charge (dual electrodynamics),

$$\overrightarrow{\nabla}\cdot\overrightarrow{j_m} + \frac{\partial\rho_m}{\partial t} = 0. \tag{26}$$

Taking the curl of, $\overrightarrow{\nabla}\times\overrightarrow{\psi} = i\overrightarrow{j_m} + i\frac{\partial\overrightarrow{\psi}}{\partial t}$ (the second set of equation (19)) and using, $\overrightarrow{\nabla}\cdot\overrightarrow{\psi} = \rho_m$ (the first set of equation (19)), we get the following differential equation

$$\square\overrightarrow{\psi} = -\overrightarrow{\nabla}\rho_m - \frac{\partial\overrightarrow{k}}{\partial t} + i\overrightarrow{\nabla}\times\overrightarrow{j_m} = \overrightarrow{S}\ (say), \tag{27}$$

where $\overrightarrow{S}$ is being introduced as a new parameter which may be recalled a field current. As such, we have established a connection between the field vector $\overrightarrow{\psi}$ and field current $\overrightarrow{S}$ in the same manner as we have established the relation (23) or (25) between the potential and current. Accordingly, we may develop the classical Lagrangian formulation in order to obtain the field equation (dual Maxwell's equations) and equation of motion for the dynamics of a dual charge (magnetic monopole) inter acing with electromagnetic field.

The Lorentz force equation of motion for a dual charge (i.e. magnetic monopole) may now be written from the duality equations (3) and (14) as

$$\frac{d\overrightarrow{p}}{d\tau} = \overrightarrow{f} = m,\quad \overrightarrow{\ddot{x}} = g(\overrightarrow{H} - \overrightarrow{v}\times\overrightarrow{E}), \tag{28}$$

where $\overrightarrow{p} = m\overrightarrow{\dot{x}} = m\overrightarrow{v}$ is the momentum, and $\overrightarrow{f}$ is a force acting on a particle of charge g, mass m and moving with the velocity $\overrightarrow{v}$ in electromagnetic fields. Equation (28) can be generalized to write it in the following four vector formulation as

$$\frac{dp^\mu}{d\tau} = f^\mu = m\ddot{x}_\mu = g\mathcal{F}^{\mu\nu}U_\nu, \tag{29}$$

where $\{U_\nu\} = \dot{x_\nu}$ is the four velocity while $\ddot{x}_\mu$ is the four-acceleration of a particle.

3 Quaternion Dual Electrodynamics

In order to obtain the field equations and equation of motion of a dual charge (i.e. a magnetic monopole), we may now revisit to the quaternionic reformulation of the dual electrodynamics. A complex quaternion (bi-quaternion) is expressed as

$$\mathsf{Q} = Q_\mu e_\mu \quad (Q_\mu \in \mathbb{C}), \tag{30}$$

where Q_μ are complex quantities for $(\mu = 0, 1, 2, 3)$ while $e_0 = 1$, e_j $(\forall j = 1, 2, 3)$ are the quaternion units satisfying the following multiplication rules,

$$e_0 e_j = e_j e_0 = e_j, \quad e_j e_k = -\delta_{jk} + \varepsilon_{jkl} e_l \quad (\forall j, k, l = 1, 2, 3), \tag{31}$$

where δ_{jk} is Kronecker delta and ε_{jkl} is the three index Levi-Civita symbol. A real or complex quaternion can be written as $\mathsf{Q} = Q_0 + Q_j e_j \Rightarrow Q_0 + \overrightarrow{Q}$ as the combination of a scalar a vector (i.e. $\mathsf{Q} = (Q_0, \overrightarrow{Q})$ with $Q_0 e_0 = Q_0$ and $\overrightarrow{Q} = Q_1 e_1 + Q_2 e_2 + Q_3 e_3$). The sum and product of two quaternions $\mathsf{a} = (a_0, \overrightarrow{a})$ and $\mathsf{b} = (b_0, \overrightarrow{b})$ are respectively defined as $(a_0, \overrightarrow{a}) + (b_0, \overrightarrow{b}) = (a_0 + b_0, \overrightarrow{a} + \overrightarrow{b})$ and $(a_0, \overrightarrow{a}) \cdot (b_0, \overrightarrow{b}) = (a_0 b_0 - \overrightarrow{a} \cdot \overrightarrow{b}, \overrightarrow{a} \times \overrightarrow{b} + a_0 \overrightarrow{b} + b_0 \overrightarrow{a})$. The norm of a quaternion is $N(\mathsf{Q}) = \mathsf{Q}\overline{\mathsf{Q}} = \overline{\mathsf{Q}}\mathsf{Q} = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2$; where $\overline{\mathsf{Q}} = (Q_0, -\overrightarrow{Q})$ (as $\overline{e_j} = -e_j$; $\overline{e_0} = e_0 = 1$) is the quaternion conjugate which follows the law of anti-automorphism i.e. $\overline{(\mathsf{pq})} = \overline{\mathsf{q}}\,\overline{\mathsf{p}}$ where p and q are two quaternions. Quaternions are associative but non-commutative in nature and form a group as well as a division ring. Quaternions have the inverse $\mathsf{Q}^{-1} = \frac{\overline{\mathsf{Q}}}{N(\mathsf{Q})}$. So, quaternions are equivalent to the four dimensional representation of our space-time. We can divide a quaternion by other quaternion resulting to a third quaternion. There is no distinction between the contravariants and covariants if we write a four vector in quaternion representation. A real quaternion may be expressed as a four vector in Euclidean space with signature $(+, +, +, +)$ while the bi-quaternion may be written in four-dimensional Minkowski space and the signature is chosen as $(-, +, +, +)$. For contraction and expansion of ranks we may use the products $\langle \mathsf{p}, \mathsf{q} \rangle_{Sc} = \frac{1}{2}(\mathsf{p}\overline{\mathsf{q}} + \mathsf{q}\overline{\mathsf{p}})$ (scalar product) and $\langle \mathsf{p}, \mathsf{q} \rangle_{Vec} = \frac{1}{2}(\mathsf{p}\overline{\mathsf{q}} - \mathsf{q}\overline{\mathsf{p}})$ (vector product) of two quaternions. As such, we may write the quaternion analogue of a space-time contravariant four vector as

$$x^\mu \mapsto \mathsf{x} = -it + \overrightarrow{x} \Leftrightarrow -it + x_1 e_1 + x_2 e_2 + x_3 e_3, \tag{32}$$

and the covariant four vector may be written in terms of analogous quaternion conjugate representation as

$$x_\mu \mapsto \bar{\mathsf{x}} = -it - \overrightarrow{x} \Leftrightarrow -it - x_1 e_1 - x_2 e_2 - x_3 e_3. \tag{33}$$

So, the quaternionic four dimensional Nabla (four differential operator) and its conjugate representations are written as

$$\boxdot = i\partial_t + e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3, \tag{34}$$

$$\overline{\boxdot} = i\partial_t - e_1 \partial_1 - e_2 \partial_2 - e_3 \partial_3, \tag{35}$$

where $\partial_t = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1} = \partial_x$, $\partial_2 = \frac{\partial}{\partial x_2} = \partial_y$, $\partial_3 = \frac{\partial}{\partial x_3} = \partial_z$. Hence, straight forwardly, we may write the following quaternionic forms respectively analogous to dual potential $\{B^\mu\} =$

$(\phi_m, \overrightarrow{B})$ and dual current $\{k^\mu\} = (\rho_m, \overrightarrow{j_m})$ associated with monopole as,

$$\mathsf{B} = i\phi_m + e_1 B_1 + e_2 B_2 + e_3 B_3, \tag{36}$$

$$\mathsf{k} = i\rho_m + e_1 j_{m1} + e_2 j_{m2} + e_3 j_{m3}. \tag{37}$$

Now operating quaternionic Nabla, given by (34), respectively on (36) and (37) and using the multiplication rules of quaternion units given by (31), we get;

$$\begin{aligned}\boxdot\mathsf{B} = {} & -(\partial_t\phi_m + \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3) \\ & - ie_1[-\partial_t B_1 - \partial_1\phi_m + i(\partial_2 B_3 - \partial_3 B_2)] \\ & - ie_2[-\partial_t B_2 - \partial_2\phi_m + i(\partial_3 B_1 - \partial_1 B_3)] \\ & - ie_3[-\partial_t B_3 - \partial_3\phi_m + i(\partial_1 B_2 - \partial_2 B_1)],\end{aligned} \tag{38}$$

and

$$\begin{aligned}\boxdot\mathsf{k} = {} & -(\partial_t\rho_m + \partial_1 j_{m1} + \partial_2 j_{m2} + \partial_3 j_{m3}) \\ & - ie_1[-\partial_t j_{m1} + \partial_1\rho_m + i(\partial_2 j_{m3} - \partial_3 j_{m2})] \\ & - ie_2[-\partial_t j_{m2} + \partial_2\rho_m + i(\partial_3 j_{m1} - \partial_1 j_{m3})] \\ & - ie_3[-\partial_t j_{m3} + \partial_3\rho_m + i(\partial_1 j_{m2} - \partial_2 j_{m1})].\end{aligned} \tag{39}$$

Comparing these equations with equations (18) and (27), we get

$$\boxdot\mathsf{B} = \psi_0 - ie_1\psi_1 - ie_2\psi_2 - ie_3\psi_3, \tag{40}$$

$$\boxdot\mathsf{k} = S_0 - ie_1 S_1 - ie_2 S_2 - ie_3 S_3, \tag{41}$$

where

$$\psi_0 = -(\partial_t\phi_m + \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3) = 0, \tag{42}$$

$$S_0 = -(\partial_t\rho_m + \partial_1 j_{m1} + \partial_2 j_{m2} + \partial_3 j_{m3}) = 0, \tag{43}$$

after applying the Lorentz gauge condition (24) for $\psi_0 = 0$ and continuity (26) for $S_0 = 0$. So, we obtain the following compact and simpler forms of quaternionic inhomogeneous wave equations for potential and current for dual charge (magnetic monopole),

$$\boxdot\mathbf{B} = \boldsymbol{\psi}, \tag{44}$$

$$\boxdot\mathsf{k} = \mathsf{S}, \tag{45}$$

where $\boldsymbol{\psi}$ and S are respectively identified as the quaternionic forms of generalized field (four-field) and field density (four-field-current) given by

$$\boldsymbol{\psi} = \psi_0 - ie_1\psi_1 - ie_2\psi_2 - ie_3\psi_3, \tag{46}$$

$$\mathsf{S} = S_0 - ie_1 S_1 - ie_2 S_2 - ie_3 S_3. \tag{47}$$

As such, the quaternion wave equations (44) and (45) are regarded as the quaternion field equations for potential and currents associated with monopoles. These quaternion field equations are invariant under quaternion and Lorentz transformations. Similarly, if we operate

(35) to (46), we get

$$\overline{\boxdot}\boldsymbol{\psi} = (-i\overrightarrow{\nabla}\cdot\overrightarrow{\psi}) - \sum_{j=1}^{3} e_j[(\partial_t\psi_j + i(\overrightarrow{\nabla}\times\overrightarrow{\psi})_j], \tag{48}$$

which is reduced to the following quaternionic wave equation on using (19) and (37) i.e.

$$\overline{\boxdot}\boldsymbol{\psi} = -i\rho_m - e_1 j_{m1} - e_2 j_{m2} - e_3 j_{m3} = -\mathsf{k} \tag{49}$$

showing the relation between fields and current and is thus analogous to the Maxwell's field equations (16) or (19) in quaternionic formulation. Now using quaternion field equations (44) and (45), we get

$$\overline{\boxdot}\,\boldsymbol{\psi} = \overline{\boxdot}(\boxdot\mathsf{B}) = (\overline{\boxdot}\boxdot)\,\mathsf{B} = -\Box\mathsf{B} = -\mathsf{k}, \tag{50}$$

$$\boxdot\,\mathbf{k} = \boxdot(-\overline{\boxdot}\boldsymbol{\psi}) = -(\boxdot\overline{\boxdot})\boldsymbol{\psi} = \Box\boldsymbol{\psi} = \mathsf{S}, \tag{51}$$

which may also be written as

$$\boxdot\overline{\boxdot}\mathsf{B} = -\Box\mathsf{B} = -\mathsf{k}, \tag{52}$$

$$\boxdot\overline{\boxdot}\boldsymbol{\psi} = -\Box\boldsymbol{\psi} = -\mathsf{S}. \tag{53}$$

Equations (52) and (53) are analogous to equations (25) and (27) in simple,compact and consistent quaternion formulation and the D'Alembertian operator $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = -\boxdot\,\overline{\boxdot} = -\overline{\boxdot}\boxdot$ is described as the negative of modulus of quaternion Nabla while if we define it in Minkowski space of signature $(-,+,+,+)$, it becomes the norm of quaternion Nabla $N(\boxdot) = \overline{\boxdot}\boxdot = \boxdot\overline{\boxdot} = \Box$. As such, we may express the quaternionic forms of the Lorentz force $\{f_\mu\}$, the velocity $\{U_\nu\}$ and the field tensor $\{\mathcal{F}_{\mu\nu}\}$ as

$$\mathfrak{F} = f_0 + e_1 f_1 + e_2 f_2 + e_3 f_3 = (f_0, \overrightarrow{f}), \tag{54}$$

$$\mathsf{U} = U_0 + e_1 U_1 + e_2 U_2 + e_3 U_3 = (U_0, \overrightarrow{U}), \tag{55}$$

$$\boldsymbol{F} = \mathcal{F}_{\mu\nu} e_\nu = (F_0, \overrightarrow{F}) = F_{\mu 0} e_0 + F_{\mu 1} e_1 + F_{\mu 2} e_2 + F_{\mu 3} e_3, \tag{56}$$

where

$$\begin{aligned}
F_0 &= F_{\mu 0} = F_{00} e_0 + F_{10} e_1 + F_{20} e_2 + F_{30} e_3, \\
F_1 &= F_{\mu 1} = F_{01} e_0 + F_{11} e_1 + F_{21} e_2 + F_{31} e_3, \\
F_2 &= F_{\mu 2} = F_{02} e_0 + F_{12} e_1 + F_{22} e_2 + F_{32} e_3, \\
F_3 &= F_{\mu 3} = F_{03} e_0 + F_{13} e_1 + F_{23} e_2 + F_{33} e_3.
\end{aligned} \tag{57}$$

As such, we see from (56) and (57) that the four components of second rank antisymmetric tensor $\mathcal{F}_{\mu\nu}$ are also quaternions. Hence, we may write the quaternion form of covariant Maxwell's equations (22) in the following manner,

$$[\boxdot,\ \boldsymbol{F}] = \mathbf{k}, \tag{58}$$

where $[\boxdot, \boldsymbol{F}]$ is the scalar product derived in the following manner,

$$
\begin{aligned}
[\boxdot, \boldsymbol{F}] &= \frac{1}{2}(\overline{\boxdot}\boldsymbol{F} + \overline{\boldsymbol{F}}\underleftarrow{\boxdot}) \\
&= \partial_0 \mathcal{F}_{\mathbf{0}} + \vec{\nabla} \cdot \vec{\mathcal{F}} = (i\rho_m, \vec{j}_m) = (\mathsf{k}_0, \vec{\mathsf{k}}) = \mathsf{k}.
\end{aligned} \tag{59}
$$

Here we the symbol ($\leftarrow$) stands the operation from right to left and ρ_m and $\vec{j}_m$ are respectively the charge and current source densities due to the dual charge given by (16, 19) and (23). Equation (58) may also be written as follows,

$$[\boxdot, \mathcal{F}_\mu] = \mathbf{k}_\mu. \tag{60}$$

Hence, we may write the analogous quaternion equation for the Lorentz force equation of motion for dual charge given by (29) as

$$g[\mathbf{U}, \boldsymbol{F}] = \Im, \tag{61}$$

where $\mathbf{U}$ and $\Im$ are respectively the quaternionic analogues of four-velocity and four force given by (54) and (55) and $[\mathbf{U}, \boldsymbol{F}]$ is the defined as the scalar product of quaternion velocity and quaternion anti-symmetric field strength as

$$
\begin{aligned}
[\mathbf{U}, \boldsymbol{F}] &= \frac{1}{2}(\overline{\mathbf{U}}\boldsymbol{F} + \overline{\boldsymbol{F}}\mathbf{U}) \\
&= U_0 \mathcal{F}_{\mathbf{0}} + \vec{u} \cdot \vec{\mathcal{F}} = (\mathsf{f}_0, \vec{\mathsf{f}}) = \Im.
\end{aligned} \tag{62}
$$

Here f_0 and $\vec{\mathsf{f}}$ are the scalar and vector components of four-force given by (29) and thus we may write (62) like (60) as

$$[\mathbf{U}, \mathcal{F}_\mu] = \mathbf{f}_\mu. \tag{63}$$

Equations (62) and (63) may also be generalized to the following quaternion form of Lorentz force equation of motion;

$$\Im = \mathbf{k} \boxdot \mathsf{B}, \tag{64}$$

where we have used the definition of the four-current density as $k_\mu = \rho_m U_\mu$ (i.e. the product of charge density and velocity) and the volume integration of charge density gives the total charge for point like monopole leading to $g = \int \rho_m d^3x$ resulting to establish the following relation between the four current and four velocity as

$$\mathbf{k}_\mu = g\, \mathbf{u}_\mu, \tag{65}$$

and the anti-symmetric field strength is replaced by the quaternionic wave equation $\boxdot \mathbf{B} = \boldsymbol{\psi}$ given by (44) to establish a connection between potential and field in quaternion formulation. Equation (64) is similar to the equation derived earlier by Waser [61, 62] for the quaternionic form of generalized Lorentz force. This has been verified here for the theories of dual electrodynamics.

4 Discussion

Starting with the concept of invariance of the electromagnetic duality in source free Maxwell's equations (1), we have shown that the Maxwell's equations (1) remain invariant under duality transformations (2, 3, 4) with the choice of four potential either to express the electromagnetic fields given by (8) or by (10). The conventional Maxwell's equations (7) follow the first choice of four-potential which describe the electromagnetic fields given by (8) and thus, violates the duality invariance due to presence of source current. Here, we have discussed the alternative choice of second four-potential which produces the electric and magnetic fields given by equations (10) to satisfy the another kind of Maxwell's equations given by equations (11) and (12). These Maxwell's equations (11) or (12) are visualized as dual Maxwell's equations as they may be obtained from the usual Maxwell's equations (7) under duality transformations (2, 3, 4). So, it is concluded that the second alternative potential is described as the dual electromagnetic potential and hence produces the electric and magnetic fields for the dynamics of dual electric charge (i.e. magnetic monopole). As such, the duality transformations (13) are established for four-potentials and four-currents of electric and magnetic charges. It has been shown that electric and magnetic charges and correspondingly the anti-symmetric electromagnetic field tensor and its dual, transform under duality transformations respectively given by (14) and (15). Accordingly, we have discussed the Maxwell's equations (19) in terms of electromagnetic field vector and developed a covariant formulation of parallel electrodynamics for dual electric charge (magnetic monopole). Subsidiary conditions like Lorentz gauge condition (24) and the continuity equation (26) are also obtained consistently. We have also developed a connection between the four potential and four currents given by field equation (25) and hence introduced a new vector parameter $\overrightarrow{S}$ as field current in the same manner to establish its relation with the electromagnetic field vector $\overrightarrow{\psi}$ given by (27). It has been concluded that like the dynamics of electric charge, we may develop the parallel dynamics of dual charge (magnetic monopole) and accordingly the equation of motion (29) has been established for Lorentz force equation of motion of magnetic monopole. Thus, we observe that either the classical electrodynamics (dynamics of electric charge-electron) or the dual electrodynamics (dynamic of magnetic monopole) suffers from the fact that the classical equations of motion are no more invariant under duality transformations. So, in order to survive the duality invariance and to symmetrize the Maxwell's equations, theories of dyons (particles carrying simultaneous existence of electric and magnetic charges) do better and the bi-quaternion formulation of generalized electromagnetic fields of dyons provide a consistent platforms to understand the existence of monopole and dyons. As such, we have discussed the bi-quaternionic formulation of dual electrodynamics given by (44, 45, 49, 52, 53, 58, 60, 61 and 64) and it has been shown that these quaternion equations are compact, simple and manifestly covariant. It has been shown that bi-quaternion reformulations of usual and dual electrodynamics describe the change of metric from $(+,-,-,-)$ to $(-,+,+,+)$. Hence it leads to the conclusion that on passing from usual Minkowski space to a quaternionic space the signature of a four-vector is changed from $(+,-,-,-)$ to $(-,+,+,+)$. Hence the mapping $(3,1) \rightarrow (1,3)$ incorporated [63–69] with complex superluminal Lorentz transformations and the quaternionic formulation for subluminal field equations are similar in nature. So, we may accordingly establish a closed connection between bi-quaternion formulation and complex superluminal transformations. The advantage in expressing the field equations in quaternionic forms is that one may directly extend the theory of subluminal to superluminal realm as well as the non commutativity of quaternion units play an important role to understand the current grand unified theories and the theories beyond the standard model of elementary particles. This formalism may also be useful to develop space-time duality between complex and quaternionic quantum mechanics such that the evolution operator for bradyons depends on time and that for tachyons depends on space. As such the bi-quaternion formulation may hope a better understanding of duality invariance as the quaternion wave equations represent their self dual nature. On the other hand, bi-quaternion analyticity provides a unified and consistent grounds for the existence of monopoles and dyons and here we have described the dual electrodynamics accordingly. It may also be concluded that the quaternion formulation be adopted in a better way to understand the explanation of the duality conjecture and supersymmetric gauge theories as the candidate for the existence of monopoles and dyons.

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